Topological Characterization and Early Detection of Bifurcations and Chaos in Complex Systems[1]

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Bifurcation and Chaos in Complex Systems

Complexity in dynamical systems:

- Nonlinearity
- High-dimensionality
- Time-varying operating conditions
- Environmental uncertainties, etc.

Bifurcations and Chaos: Critical transitions characterized by changes in topological features

- **Causes**: Changes (parametric/non-parametric) in system
- **Effects**: System anomalies, undesirable performance and failures
- **Mitigation**: Early detection of transitions and take proactive actions

Example 1: Duffing Oscillator

Duffing Oscillator

$$\ddot{y}(t) + \delta\dot{y}(t) + \alpha y(t) + \beta y^3(t) = \gamma\cos(\omega t) \qquad (1)$$

Where $y(t)$: displacement at time t

δ: controls damping

α: controls linear stiffness

β: controls non-linearity

γ: amplitude of periodic driving force

ω: angular frequency of the driving force

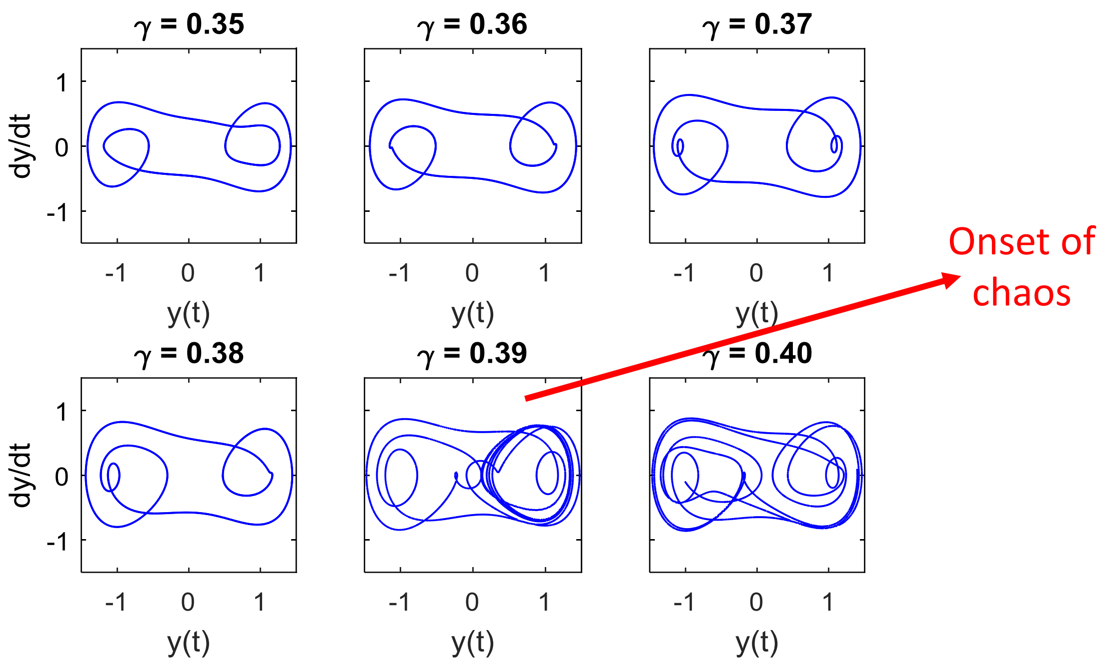


Fig.1. Phase portrait plots for different γ values

Example 2: Logistic Map

2. Logistic Map

$$f\big(x(n)\big) = x(n+1) = r * x(n)\big(1 - x(n)\big) \quad , x(n) \in [0,1] \qquad (2)$$

Where $x(n)$: ratio of population at time instance n to the maximum possible population

r: growth rate parameter

Period-Doubling cascade

Onset of chaos

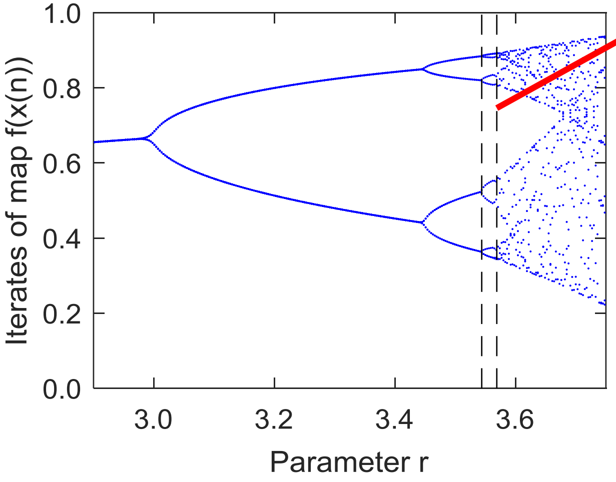


Fig.2. Period-doubling cascade

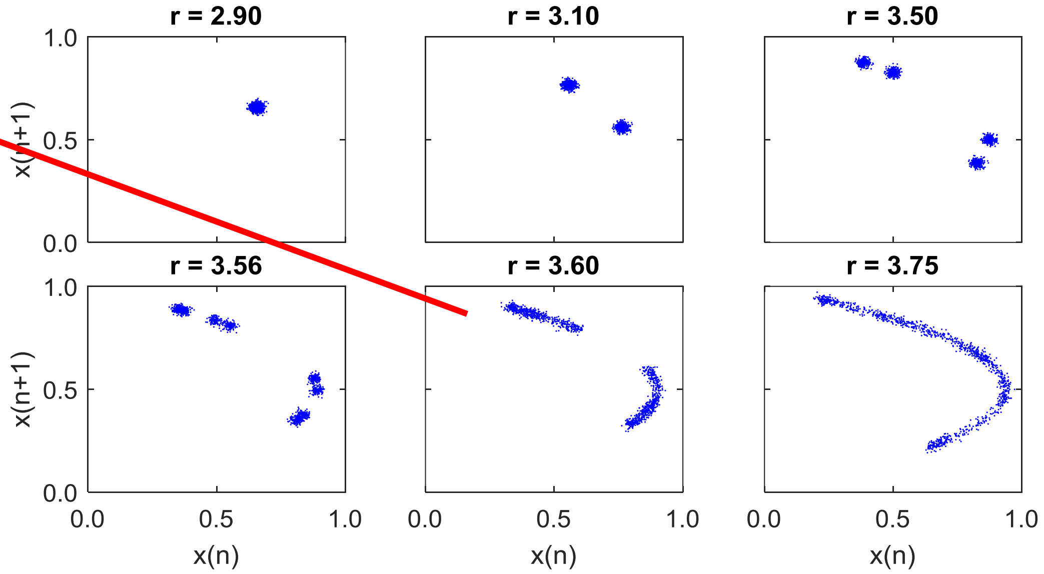


Fig.3. Phase-space reconstruction plots for different r values with embedding delay $= 1$ and dimension $= 2$

Research Objective and Existing Methods

Goal: To develop a mathematical framework of topological data analysis for:

- Early detection of bifurcations and chaos
- Understanding their topological characteristics

Existing approaches:

I. **Bendixson-Dulac criterion and Poincare-Bendixson[2] criterion**
   - Features: Detects the presence of limit cycles
   - Limitations: Not applicable for high-dimensional systems (D>2)

II. **Harmonic Balance and Describing Functions[3]**
   - Features: Provide an approximate estimate of the size of limit cycles
   - Limitations: Linearization might fail for systems with higher harmonics of nonlinearity

III. **Data-driven methods such as recurrence plots[4], correlation sum analysis[5], Lyapunov exponents[6], permutation entropy[7] and symbolic dynamics[8]**
   - Features: Detection of anomalies or changes in system behavior
   - Limitations: No topological understanding to distinguish between the system behaviors before and after transitions

Advantages of Persistent Homology

Research Gaps
Topological information such as the presence of sub-cycles, their positions and sizes not known
⇒ No topological insights for early detection of bifurcation and chaos

Benefits of persistent homology

- Extraction of topological features: number of relevant k-dimensional holes, their positions, sizes, and lifetimes
- Early detection of bifurcations and chaos by tracking the evolution of the above features
- Robustness to noise and uncertainties and applicability on high-dimensional data.

Persistent Homology[9]

Used for computing topological features (or Betti numbers) of a space at different spatial resolutions.

Some mathematical preliminaries

***k*-Simplex:** k-dimensional polytope which is a convex hull of its $k+1$ vertices.

Simplicial complex[10] ($\mathcal{R}$): Set of simplices such that:

- Any face of a simplex from $\mathcal{R}$ is also in $\mathcal{R}$,
- The intersection of any two simplices $\sigma_1, \sigma_2 \in \mathcal{R}$ is either $\emptyset$ or a face of both σ_1 and σ_2.

Examples: Vietoris-Rips (VR) complex, Witness complex, etc.

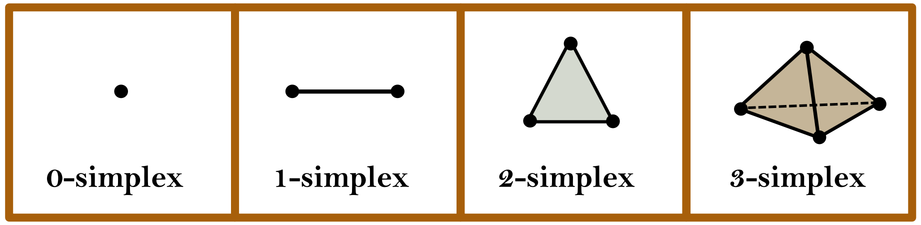
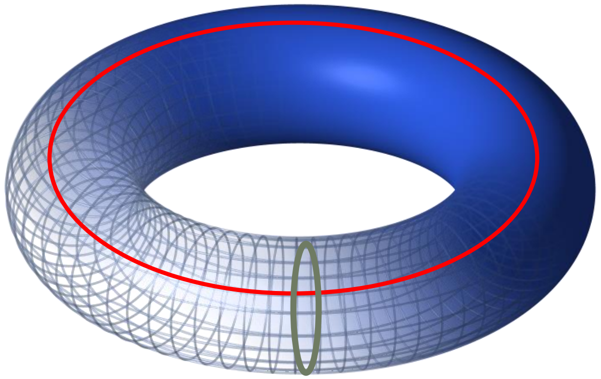


Fig.4. Low-dimensional k-simplices for $0 \leq k \leq 3$

Betti Numbers[9]

β_k(k-D Holes)	1-D	2-D	3-D
β_0(0-D Holes)	No. of connected components	No. of connected components	No. of connected components
β_1(1-D Holes)	0	No. of circular holes	No. of circular holes
β_2(2-D Holes)	0	0	No. of voids
β_3(3-D Holes)	0	0	0



$$\beta_0 = 1$$
$$\beta_1 = 2$$
$$\beta_2 = 1$$

Fig.5. Betti numbers for a 3-D torus

Betti Numbers: Example

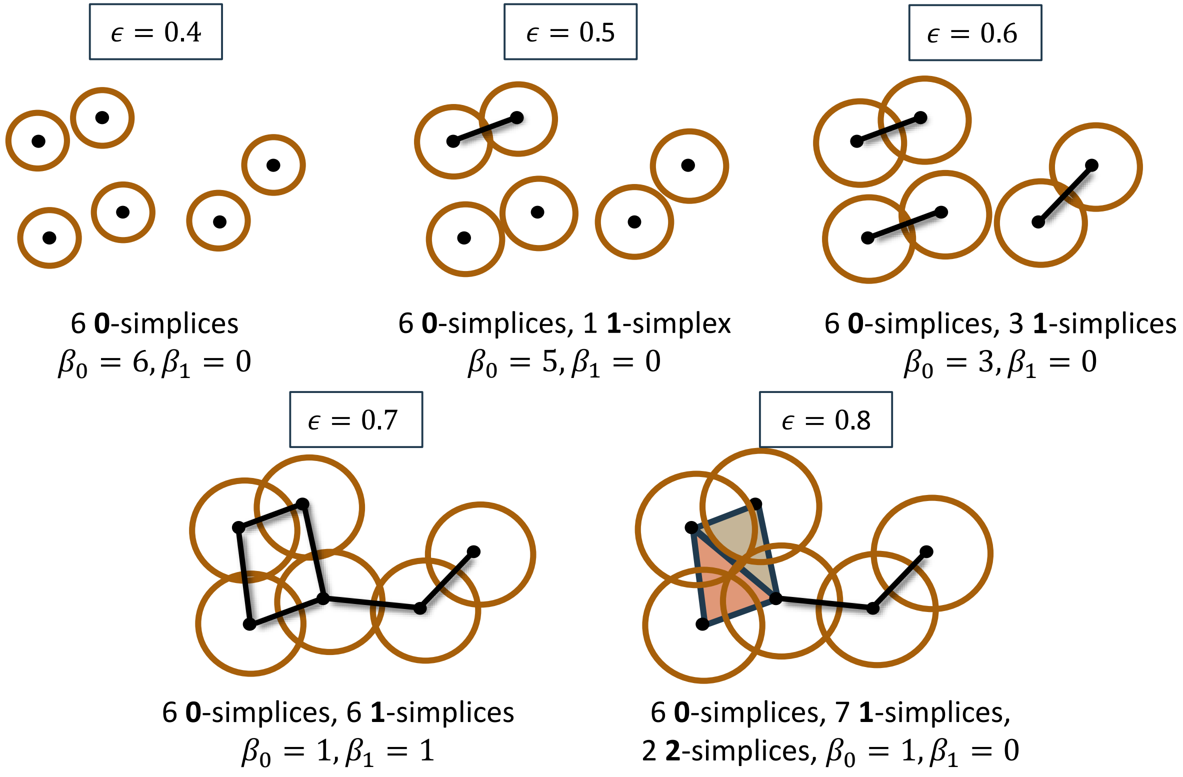


Fig.6. Example of Vietoris-Rips complexes and corresponding Betti numbers for different scale parameters ϵ.

Persistent Homology

Homology Groups[11]: Computed from simplicial complexes and provide information of Betti numbers.

How to get the optimal scale parameter ϵ?

- Small ϵ: VR complex containing discrete data points
- Large ϵ: High-dimensional simplex

Persistent Homology: Computes simplicial complexes for a range of ϵ values.

Key idea: To examine the homology of these iterated complexes (called as *filtration*).

Persistent homology groups provide lifetime information of each k-dimensional hole

$$\Delta_k = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0, u \leq v\} \quad (3)$$

where u: birth scale of a hole,

v: death scale of a hole

Persistent Intervals

Persistent Intervals: Represent the evolution of holes (increase and decrease in Betti numbers)

Most persistent/longest interval ⇒ True Betti numbers of a topological space

Visualization of persistent intervals**: Barcode Plots[12] and Persistence Diagrams**

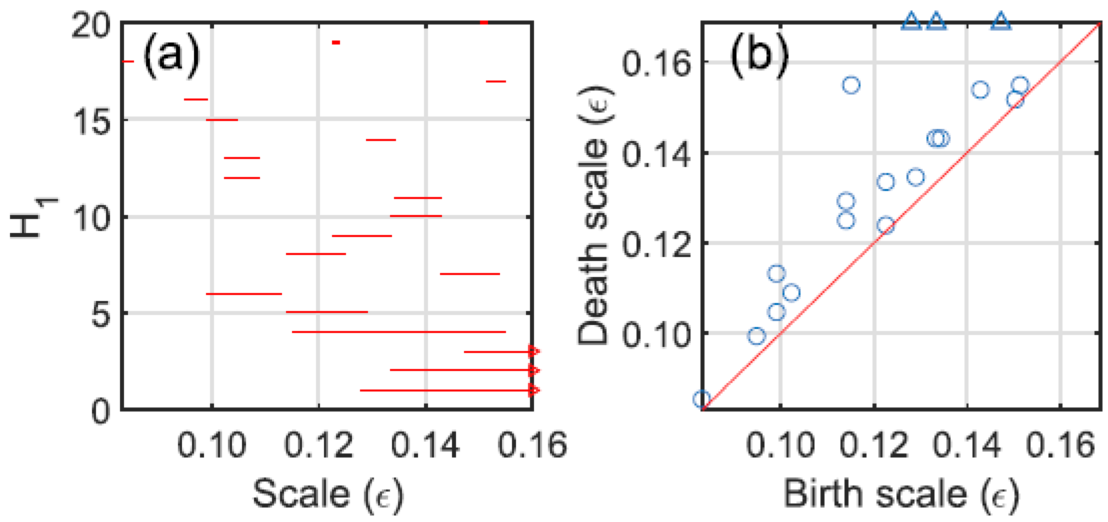


Fig.7. Illustration of persistent homology features: (a) barcode plot for 1-D holes and (b) persistence diagram.

Proposed Topological Features

I. **No. of relevant k-D holes (nrel_k)[12] for $k \geq 0$**: Number of holes with lifetime greater than $\theta * ML_k$ where $\theta \in (0, 1)$ and ML_k is the maximum lifetime of k-D holes

$$ML_k = \max_{1 \leq i \leq \Delta_k} L_i \tag{4}$$

$$L_i = v_i - u_i \tag{5}$$

II. **Average lifetime of k-D holes[12] (avg_k):**

$$\mathrm{avg_k} = \frac{1}{|\Delta_k|} \sum_{i=1}^{|\Delta_k|} L_i \tag{6}$$

III. **Expected orbit period (K_{orbit}):**

$$\mathrm{K_{orbit}} = \left\lfloor \frac{S_1}{S} \right\rfloor \tag{6}$$

where S_1= number of simplices for a stable system with period-1 orbits converging to 1 fixed point

S = number of simplices for a given system

Proposed Topological Features

IV. Diameter of a hole (D_i): For a 1-D hole i in M-dimensional space containing N vertices

$$D_i = \max_{1 \le p,q \le N} \left\| \boldsymbol{x_p} - \boldsymbol{x_q} \right\| \tag{8}$$

where $\boldsymbol{x_p}, \boldsymbol{x_q} \in \mathbb{R}^M$ belong to the hole.

V. Maximum diameter of k-D holes ($\mathrm{maxD_k}$)

$$\mathrm{maxD_k} = \max_{1 \le i \le |\Delta_k|} D_i \tag{9}$$

VI. Maximum Distance between k-D holes ($\mathrm{maxDist_k}$):

$$\mathrm{maxDist_k} = \max_{1 \le i,j \le |\Delta_k|} dist_k^{ij} \tag{10}$$

$$dist_k^{ij} = \left\| \boldsymbol{xc_i} - \boldsymbol{xc_j} \right\| \tag{11}$$

$$xc_j^m = \frac{1}{N} \sum_{j=1}^{N} x_j^m \; for \; m = 1,2, \dots M \tag{12}$$

where $\boldsymbol{xc_i} = \left(xc_i^1, xc_i^2, \dots, xc_i^M \right)$ is the center of i^{th} hole

$\boldsymbol{x_j} \in \mathbb{R}^M$ is a vertex of the hole, $dist_k^{ij}$ is the distance between holes i and j.

Results: Duffing Oscillator

Simulations:

$$\ddot{y}(t) + \delta\dot{y}(t) + \alpha y(t) + \beta y^3(t) = \gamma \cos(\omega t) \tag{13}$$

- $\delta = 0.3, \alpha = 1, \beta = 1, \gamma \in [0.35, 0.38], \omega = 1.2$
- $y(0) = 0, \dot{y}(0) = 0$
- Additive White Gaussian Noise (AWGN) of 30dB signal-to-noise ratio (SNR) is added and denoised through Wavelet filtering
- Persistent homology computations: Javaplex[13] toolbox in MATLAB
- Witness complex is used for persistent homology with $\epsilon \in [0, 0.001]$
- Features used: $nrel_1$ with $\theta = 0.7$, avg_1 and D_i

Results: Duffing Oscillator

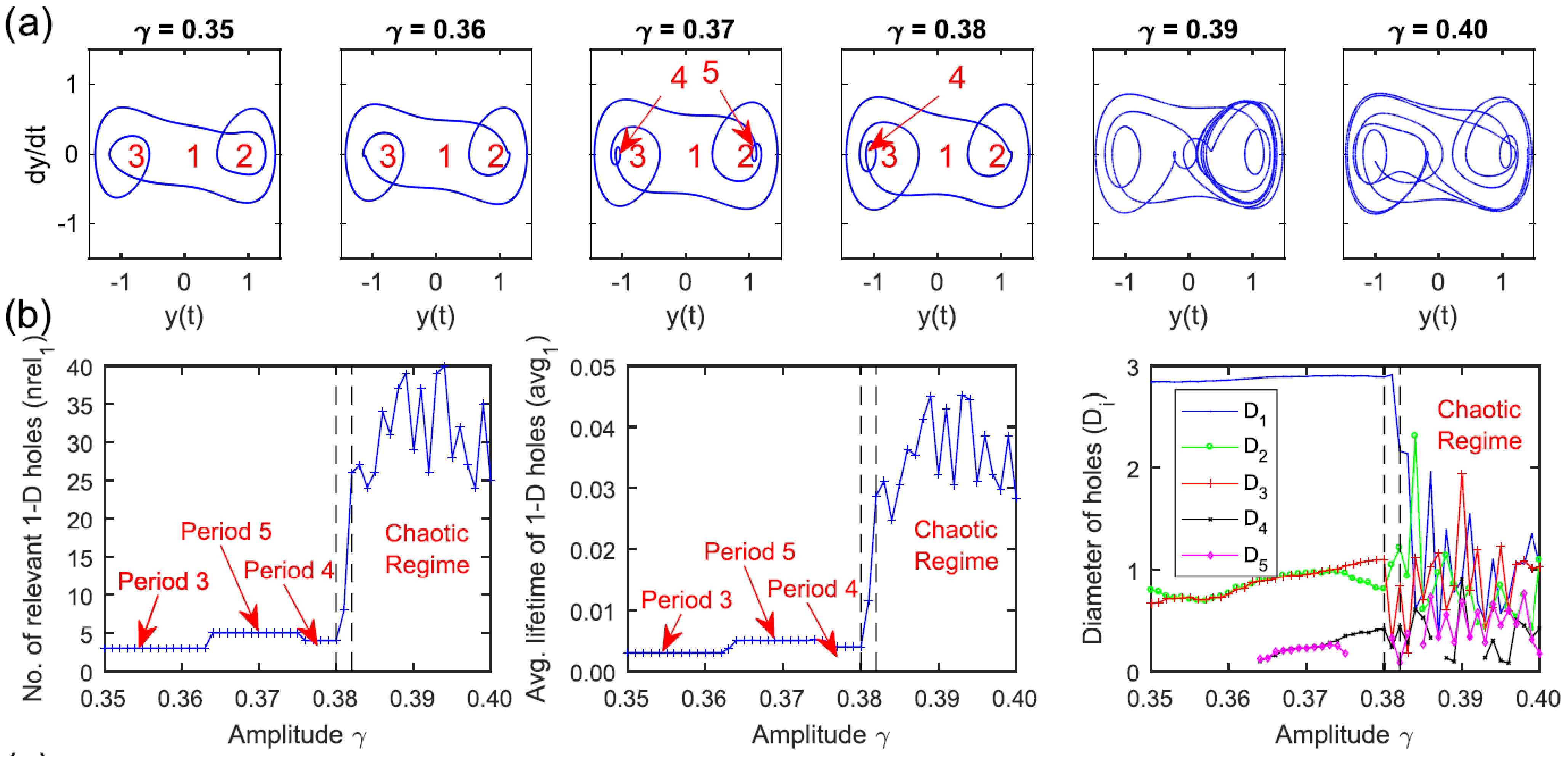


Fig.8. Analysis of the Duffing oscillator: (a) phase portrait plots for different γ values, (b) proposed topological features

Results: Logistic Map

Simulations:

$$f\big(x(n)\big) = x(n+1) = r * x(n)\big(1 - x(n)\big) \quad , x(n) \in [0, 1] \qquad (14)$$

- $r \in [2.5, 3.75]$
- AWGN with 30dB SNR is added to the data
- Phase-space reconstruction using Taken's theorem[13] is applied to generate point cloud data with delay = 1 and embedding dimension = 2.
- Persistent homology computations: Javaplex toolbox in MATLAB
- VR complex filtration is used with $\epsilon \in [0, 0.01]$
- Features used: K_{orbit} with S_1= number of simplices at $r = 2.9$, avg_0 and $maxDist_0$

Results: Logistic Map

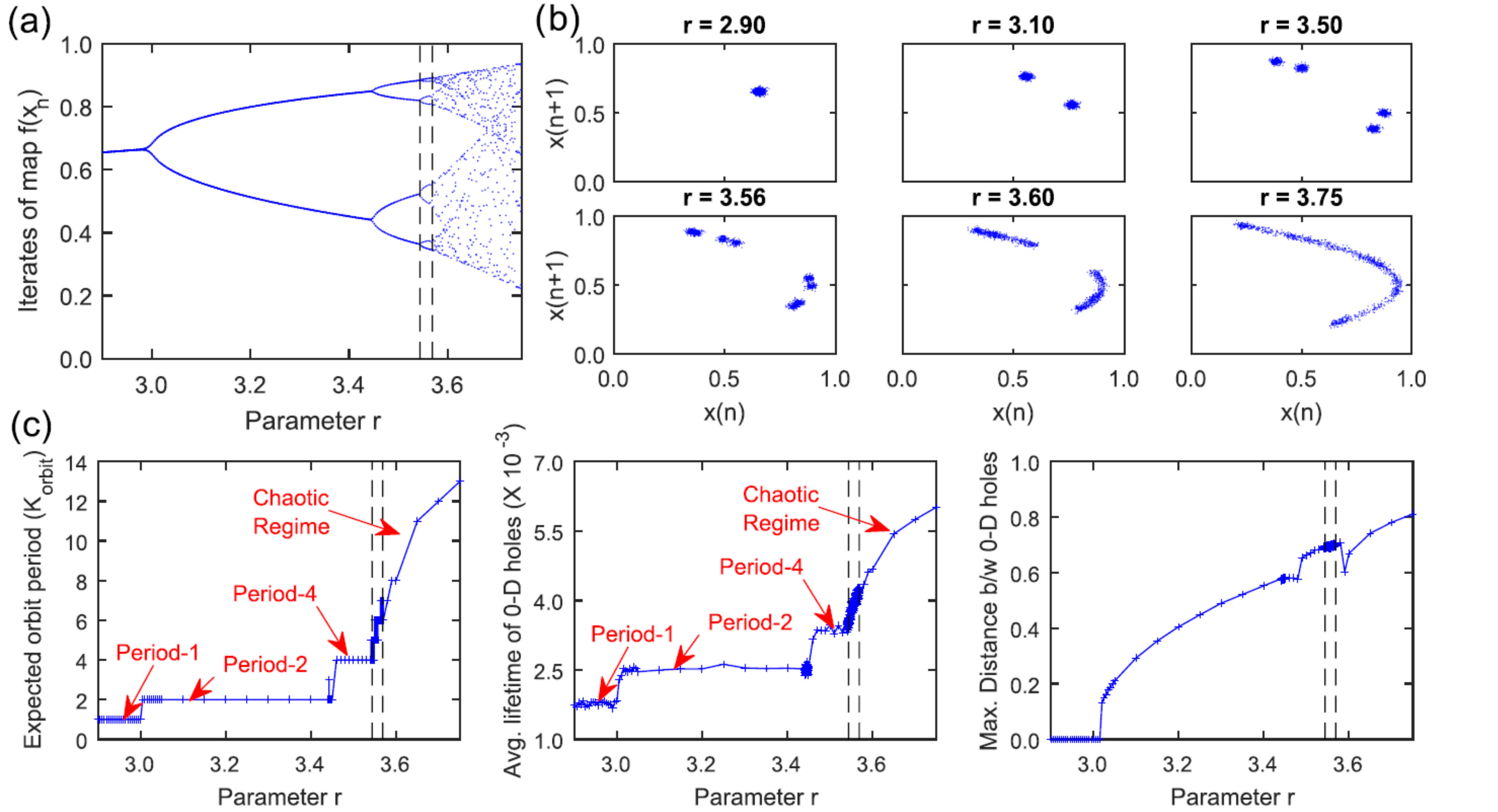


Fig.9. Analysis of the Logistic map: (a) period-doubling cascade, (b) phase space reconstruction plots for different r values and (c) proposed topological features

Conclusion and Future Work

- An approach for topological characterization of complex systems is presented
- Early detection of bifurcations and chaos is achieved
- Validation on Duffing Oscillator and Logistic Map

Future work:

- Application of the proposed features for
  - anomaly detection in other real world time series data
  - epileptic seizure detection, behavior prediction and fault detection

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